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ABSTRACT

By assuming only crossing symmetry and analyticity in the 3 body scattering amplitude, we prove that the ratio of the inclusive cross section for the process $a+b \to \bar{c}+X$ to the semi-inclusive cross section for the process $a+b \to \bar{c}+X_s$, where X_s is any subset of X, is a function of M^2, the missing mass squared of X (and X_s) only. We show that the ratio does not depend on s or t. Several related lemmas are also proved. The results are valid at all energies and for all interactions possessing analyticity and crossing symmetry.

Consider the scattering process described by Figure 1(a). The cross section for the scattering process is a function of the variables M^2, s and t and may be written as

$$B(M^2,s,t) = K \sum_X \langle abc|R^\dagger|X\rangle\langle X|R|abc\rangle \qquad (1)$$

where R is the reaction matrix, $|abc\rangle$ is a ket denoting the initial state and $|X\rangle$ is a ket denoting the allowed set of final states abc can scatter into. K contains kinematical flux factors and will not concern us further. We prefer to use the discrete summation notation rather than phase space integration for the sake of clarity. In equation (1), $\Sigma|X\rangle\langle X|$ is an operator acting on the Hilbert space of $|abc\rangle$. Since $|X\rangle$ contains only those states with effective mass M, the operator is not unity. Let us note that owing to Lorentz invariance, S is a function of M^2 only and not of s and t.

The usual laws of operator algebra apply to S. In particular, multiplication of S by a scalar yields another operator.

The cross section for abc scattering into a subset X_s of X may be written as

$$C(M^2,s,t) = K \sum_{X_s} \langle abc|R^\dagger|X_s\rangle\langle X_s|R|abc\rangle \qquad (2)$$

X_s is any subset of X, e.g. X_s has a fixed multiplicity, X_s does not contain any baryons etc. $\Sigma|X_s\rangle\langle X_s|$ is another operator which we will denote by S_1. S_1 is a function of M^2 only. From equations 1 and 2, we get

$$\frac{g(M^2,s,t)}{B(M^2,s,t)}\Sigma\langle abc|R^\dagger|X\rangle\langle X|R|abc\rangle = \frac{g(M^2,s,t)}{C(M^2,s,t)}\Sigma\langle abc|R^\dagger|X_s\rangle\langle X_s|R|abc\rangle$$

$$= g(M^2,s,t) \qquad (3)$$

where g is any single valued, differentiable, continuous function of M^2,s,t. Equation 3 establishes a correspondence between the operators $\frac{g(M^2,s,t)S}{B(M^2,s,t)}$ and $\frac{g(M^2,s,t)S_1}{C(M^2,s,t)}$, all other quantities being identical on both sides of the equation. Note that we are not equating the two operators, only establishing a correspondence.i.e. to every operator belonging to the set $(g/B)S$, there exists a corresponding operator $(g/C)S_1$ which acting on the same set of states yields the same function $g(M^2,s,t)$.

Since the cross sections $B(M^2,s,t)$ and $C(M^2,s,t)$ are single valued functions of M^2, s and t, and since for a given M^2, one can only construct one S and one S_1, there exist, for an arbitrary function g, neighbourhoods in M^2,s,t

space where the correspondence is a one-to-one correspondence. Figure 2 shows the correspondence for 3 neighbouring points in M^2,s,t space with the same M^2 and s. The operators shown are discrete .For strict rigour, one can normalize all the states in a box in space-time to obtain denumerable, discrete states and let the dimensions of the box become large.

We may write the local isomorphism symbolically as

$$\frac{g(M^2,s,t)S}{B(M^2,s,t)} \longleftrightarrow \frac{g(M^2,s,t)S_1}{C(M^2,s,t)} \tag{4}$$

For the present, we defer considering the family of lines in M^2,s,t space along which either $(g/B)S$ or $(g/C)S_1$ remain constant. Correspondence (4) holds true for a wide class of arbitrary functions $g(M^2,s,t)$. We assume that $B(M^2,s,t)$ is a well-enough behaved function to belong to that class. This implies

$$S \longleftrightarrow \frac{B(M^2,s,t)S_1}{C(M^2,s,t)} \tag{5}$$

Since S and S_1 are functions of M^2 only, this can only be true if the ratio $B(M^2,s,t)/C(M^2,s,t)$ is a function of M^2

only and not of[2] s and t. Otherwise for each S. there will correspond an infinite number of operators in contradiction to 5. QED. Note once again that at no point in the proof have we equated the elements of one set of operators with the elements of any other. (see Figure 2.)

$$\text{Therefore} \quad \frac{B(M^2,s,t)}{C(M^2,s,t)} = \beta(M^2) \qquad (6)$$

Equation (6) implies that the family of lines in M^2,s,t space along which (g/B)S remains constant is the same family along which $(g/C)S_1$ remains constant. Thus (6) implies a one-to-one correspondence throughout the M^2,s,t space. We note in passing that equation 6 is equivalent to factorization of the three body scattering process into a formation vertex and a decay vertex as shown in Figure 1(b). Following Mueller,[3] one may now continue t to the region for the process $a+b \to \bar{c}+X$. This is permissible, if the amplitude possesses analyticity and crossing symmetry. This yields the relation

$$\frac{f(ab \to \bar{c}+X_S)}{f(ab \to \bar{c}+X)} = \alpha(M^2) \equiv 1/\beta(M^2) \qquad (7)$$

, where f denotes the invariant inclusive cross-section $E\frac{d^3\sigma}{dp^3}$. One can continue both s and u (the sub-energy of the combination ac) to yield

$$\frac{f(ab\to\bar{c}+X_s)}{f(ab\to\bar{c}+X)} = \frac{f(ac\to\bar{b}+X_s)}{f(ac\to\bar{b}+X)} = \frac{f(bc\to\bar{a}+X_s)}{f(bc\to\bar{a}+X)} = \alpha(M^2)$$

(8)

This implies that the three crossed channels are governed by the same function α, which depends only on M^2 and the subset chosen. In particular, at any fixed M^2, the t distributions of the subset and the whole set must have exactly the same shape; conversely at various t intervals the ratios of the M^2 distributions of the subset to the set should be the same function α, of M^2, irrespective of the t interval. The relations 8 are easily verifiable by experiment.

1st Lemma:- The mean recoiling multiplicity $\langle n_X \rangle$ defined as $\Sigma n_i f_i/f$,(where f denotes the inclusive cross section for the process $ab \to \bar{c}+X$ and f_i denotes the semi-inclusive cross section for the process $ab \to \bar{c} + n_i$ charged particles), is independent of s and t and is only a function of M^2. For,

$$\langle n_X \rangle = \Sigma\ n_i f_i/f = \Sigma\ n_i \alpha_i (M^2) = \text{function of } M^2 \qquad (9)$$

Keeping M^2 fixed, one may integrate over t to yield the well known experimental result [4] that the mean recoiling multiplicity (integrated over t) is a function of M^2 only and not of s. The same function , again governs the three crossed channels.

2^{nd} Lemma:- We may now employ a similar approach towards the reactions $\bar{p}p \to \pi^{\pm}+X$ and $\bar{p}p \to \pi^{\pm}+X_s$. Let B_1 denote $f(\bar{p}p \to \pi^{+}+X)$ and C_1 denote $f(\bar{p}p \to \pi^{+}+X_s)$, and B_2 and C_2 denote $f(\bar{p}p \to \pi^{-}+Y)$ and $f(\bar{p}p \to \pi^{-}+Y_s)$ respectively. Then we obtain the operator correspondences,

$$S \equiv \Sigma \quad |X\rangle\langle X| \longleftrightarrow \frac{B_1}{C_1} \cdot \Sigma \quad |X_s\rangle\langle X_s| \equiv \beta(M^2)S_1 \qquad (10)$$

$$S' \equiv \Sigma \quad |Y\rangle\langle Y| \longleftrightarrow \frac{B_2}{C_2} \cdot \Sigma \quad |Y_s\rangle\langle Y_s| \equiv \beta'(M^2)S_1' \qquad (11)$$

Note now that the operator S contains all the charge conjugate states of S´ and that S_1 contains all the charge conjugate states of S_1'. Hence by charge symmetry it follows that $\beta(M^2) \equiv \beta'(M^2)$.

$$\text{i.e.} \quad \frac{C_1(M^2,s,t)}{B_1(M^2,s,t)} = \frac{C_2(M^2,s,t)}{B_2(M^2,s,t)} = \alpha(M^2) \qquad (12)$$

This leads to $\dfrac{C_1-C_2}{C_1+C_2} = \dfrac{B_1-B_2}{B_1+B_2}$, a relation arrived at heuristically before for annihilations.[5] Here we prove it for the general case where X_s is <u>any</u> subset of X. The result is generalisable to any initial state(e.g.$\bar{p}p$) that is charge conjugate with itself. We emphasise that the variable t is defined for π^+ and π^- analogously, as the momentum transfer squared from the target proton. Thus equation (12) is in addition to the usual charge symmetry relations $C_1(M^2,s,t) = C_2(M^2,s,u)$ and $B_1(M^2,s,t) = B_2(M^2,s,u)$ which arise from reflection symmetry in the center of mass .

3rd Lemma:- The second lemma is easily generalisable to processes of the type $ab \to c+X$ and $de \to f+Y$ where the set abc is the symmetry conjugate of the set def. If in addition, de is the symmetry conjugate of ab, the predictions can be tested in the same experiment. Otherwise, two separate experiments are required. As an example, consider the symmetry operation under which the third component of isospin is reflected. Let the two reactions under consideration be $np \to \pi^+ +X$ and $np \to \pi^- +Y$. Under the reflection of the third component of isospin, $n \to p$ and $\pi^+ \to \pi^-$ and vice versa. One would therefore, expect equation (12) to hold for the charged pion spectra for np interactions, the quantities being defined analogously, and the same experiment can be used to test their validity.

4th Lemma:- Under the assumption that the analytically continued n body scattering amplitude yields the cross section for the process $ab \to$ n-2 particles +X. we get the generalisation of equation 7 for multiparticle inclusive reactions, i.e. the ratio of the cross section for any subset to the whole is a function of M^2 only and not of other kinematical variables and that the same function applies to all the crossed channels associated with it.

By lemma 1, the three crossed channels $\bar{p}p \rightarrow \pi^-+X$, $\pi^+p \rightarrow p+X$, and $\pi^+\bar{p} \rightarrow \bar{p}+X$ have the same mean recoiling multiplicity for any given M^2. From equation (12), $\bar{p}p \rightarrow \pi^++X$ and $\bar{p}p \rightarrow \pi^-+X$ have the same $\alpha_i(M^2)$, where i is the multiplicity of the recoiling system. Therefore, the three crossed channels $\bar{p}p \rightarrow \pi^++X$, $\pi^-p \rightarrow p+X$ and $\pi^-\bar{p} \rightarrow \bar{p}+X$ have the same mean recoiling multiplicity distribution as the three mentioned before. Also, the annihilation component in $\bar{p}p$ interactions is difficult to measure at high energies. However, one can determine α for annihilations from low energy data, extrapolate it to higher values of M^2 and using equation 7, work out the full inclusive pion spectra from annihilations at high energy!

From Figure 1(b), one can deduce that the function $\alpha_i(M^2)$ is the probability of the intermediate state to decay into i charged particles. Consider the channel $pp \rightarrow p+X$. The overall distribution $\frac{d\sigma}{dM^2}$ for this channel peaks at low M^2 values near the threshold, this being attributed to diffraction dissociation. From equation (7), $\left(\frac{d\sigma}{dM^2}\right)_{i\ \text{prongs}} = \alpha_{i-1}(M^2)\left(\frac{d\sigma}{dM^2}\right)_{\text{overall}}$. Since $\frac{d\sigma}{dM^2}$ peaks at low M^2 values, we would expect the main contribution to come from subsets for which $\alpha(M^2)$ is large at low M^2 values. But this is so for low values of the recoiling multiplicity. Hence we would expect the diffractive peak to occur predominantly for low primary multiplicities, another experimentally observed fact.[*] Conversely, $\frac{d\sigma}{dM^2}$ for channels such as $pp \to \pi^- + X$, which peaks at high M^2 values, would be more evenly shared between the multiplicities.

To conclude, our main results are

(1) we have proved, from very general assumptions, that the ratio of the invariant inclusive cross section at any point in phase space to the invariant semi-inclusive cross section (where semi-inclusive denotes any subset of the recoiling system) is a function of M^2 only.

(2) For a given M^2, the ratio obtained from one channel should equal the ratio obtained from the two other cross channels. i.e there exists a single function $\alpha(M^2)$ that governs the three channels connected by crossing.

(3) The mean recoiling multiplicity is a function of M^2 only and not of s or t.

(4) Applying the theorem to $\bar{p}p$,we obtain results arrived at heuristically before.

(5)The theorem is generalisable to multiparticle inclusive reactions.

(6)The theorem is applicable to a wide class of interactions, strong, electromagnetic and perhaps weak.

(7)It has not escaped the author's notice that the relations proved here severely restrict the possible shapes of particle spectra perhaps to the point of excluding all others except the observed ones.

REFERENCES

[1] This is known in pure mathematics as an isomorphism between the two sets of operators.

[2] Note that application of this approach to two body scattering leads to the obvious and trivial result that the ratio of the cross section of a subset to the total cross section should be a function of s only.

[3] A.H.Mueller, Phys. Rev. D2, 2963 (1970).

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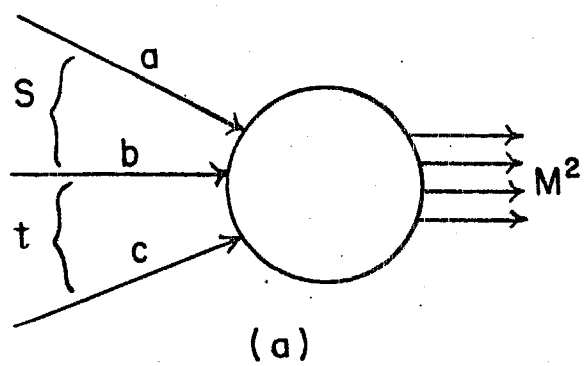
[5] R.Raja, FNAL Pub 76/99 , submitted to Phys. Rev. Lett.



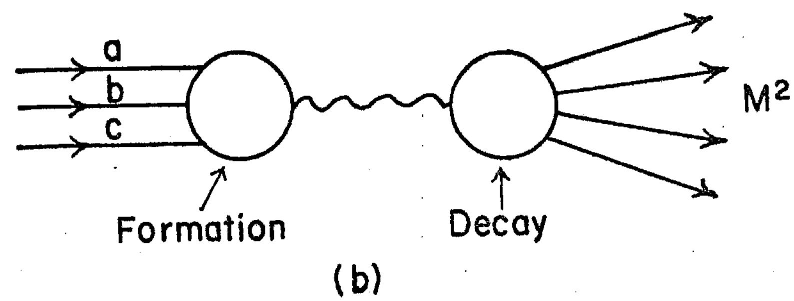


Fig. 1(a). The diagram for 3-body scattering, s and t denote the subenergies of the particle combinations shown. The subenergy of the third combination, u, is determined once the other variables are given.

Fig. 1(b). Diagram for 3 body scattering illustrating factorization.

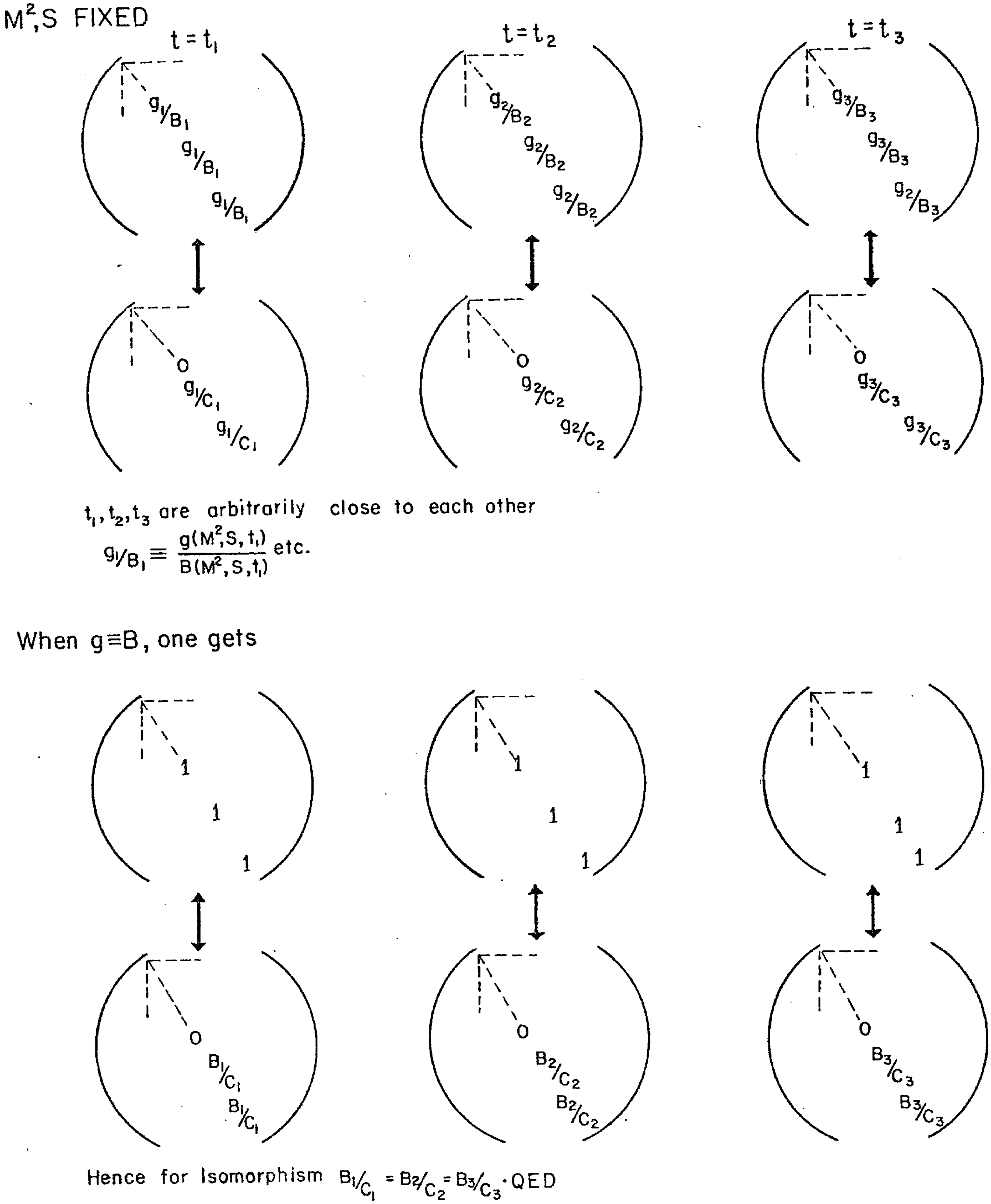


Fig. 2. Diagram showing local isomorphism between discrete operators in M^2, s, t space. The operators are diagonal. Only non-zero elements are in general shown. As M^2 changes, the non-zero diagonal elements move along the diagonal. The figure shows the isomorphism for fixed M^2, s. One can repeat the argument for fixed M^2, t.